On behaviour of solutions of system of linear differential equations*

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Abstract. *The present paper deals with the existence, behaviour and approximation of some solutions of the nonhomogeneous system of linear differential equations with functional coefficients. The behaviour of solutions in the neighbourhood of an arbitrary or respective curve is considered. The qualitative analysis theory of differential equations and the topological retraction method are used.*

Key words: *system of linear differential equations, qualitative analysis, behaviour of solutions, approximation of solutions, asymptotic solutions.*

Sažetak. **O ponašanju rješenja sustava linearnih diferencijalnih jednadžbi.** *Rad se odnosi na egzistenciju, ponašanje i aproksimaciju nekih rješenja nehomogenog sistema linearnih diferencijalnih jednadžbi s funkcionalnim koeficijentima. Promatra se ponašanje rješenja u okolini proizvoljne ili odredjene krivulje. Koristi se teorija kvalitativne analize diferencijalnih jednadžbi i topološka metoda retrakcije.*

Ključne riječi: *sustav linearnih diferencijalnih jednadžbi, kvalitativna analiza, ponašanje rješenja, aproksimacija rješenja, asimptotska rješenja.*

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1. Introduction

Let us consider a system of equations

$$x' = A(t)\,x + f(t)\,, \tag{1}$$

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where $t \in I = (a, \infty)$, $a \in \mathbb{R}$; $x \in \mathbb{R}^n, n \geq 2$; the matrix-function $A(t) = (a_{ij}(t))_{i,j=1,\dots n}$ is real and continuous on interval I; the vector-function $f(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$ is real and continuous on interval I.

Let

$$\Gamma = \{(x,t) \in \Omega : x = \varphi(t)\},$$

where $\Omega = \mathbb{R}^n \times I$, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$, $\varphi_i \in C^1(I, \mathbb{R})$, be an arbitrary curve.

The system (1) has been studied by various authors, e.g., [2, 4, 6, 8]. In [8] the system (1) is considered with respect to the set

$$\sigma = \{(x,t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t),\ i = 1, \dots, n\},$$

where $r_i \in C(I, \mathbb{R}^+)$, $i = 1, \dots, n$, $\mathbb{R}^+ = (0, \infty)$. In this paper we shall consider the behaviour of integral curves of (1) with respect to the set

$$\omega = \left\{(x,t) \in \Omega : \sum_{i=1}^{n} \alpha_i^2(t) [x_i - \varphi_i(t)]^2 < 1\right\}$$

(the neighbourhood of curve Γ), where $\alpha_i \in C^1(I, \mathbb{R}^+)$, $i = 1, \dots, n$. The obtained results are new and independent of known results.

2. Preliminaries

To obtain our results we need the following results concerning the applicability of the qualitative analysis theory and the topological retraction method of T. Ważewski (see, e.g., [2,3,5,8,9])

We shall consider the behaviour of the integral curves $(x(t), t)$, $t \in I$ of the system (1) with respect to the set ω. The boundary surface of ω is

$$W = \left\{(x,t) \in \Omega : H(x,t) := \sum_{i=1}^{n} \alpha_i^2(t) [x_i - \varphi_i(t)]^2 - 1 = 0\right\}.$$

Let us denote the tangent vector field to an integral curve $(x(t), t)$ of (1) by X. The vector ∇H is the external normal on surface W. We have

$$\begin{aligned} X(x,t) &= \left(\sum_{j=1}^{n} a_{1j}(t) x_j + f_1(t), \dots, \sum_{j=1}^{n} a_{nj}(t) x_j + f_n(t), 1\right), \\ \frac{1}{2}\nabla H(x,t) &= \left(\begin{array}{c} \alpha_1^2(t)[x_1 - \varphi_1(t)], \dots, \alpha_n^2(t)[x_n - \varphi_n(t)], \\ \sum\limits_{i=1}^{n} \left\{\alpha_i(t)\alpha_i'(t)[x_i - \varphi_i(t)]^2 - \alpha_i^2(t)\varphi_i'(t)[x_i - \varphi_i(t)]\right\} \end{array}\right). \end{aligned}$$

By means of scalar product $P = \left(\frac{1}{2}\nabla H, X\right)$ on W we shall establish the behaviour of integral curves of (1) with respect to the set ω.

Let us designate with $S^n(I)$ a class of solutions defined on I which depends on n parameters. We shall simply say that the class of solutions $S^n(I)$ belongs to the set ω if graphs of functions in $S^n(I)$ are contained in ω. In that case we shall write $S^n(I) \subset \omega$.

The results of this paper are based on the following *Lemmas.*

Lemma 1. *If, for the system (1), the scalar product $P < 0$ on W, then the system (1) has a class of solutions $S^n(I)$ belonging to the set ω for every $t \in I$, i.e., $S^n(I) \subset \omega$.*

Proof. The condition $P < 0$ on W means that integral curves of the system (1) pass through points of the set W entering ω, i.e., the set W is a set of points of strict entrance of integral curves of (1) with respect to the sets Ω and ω (see, e.g., [3, 5]). Hence, all integral curves passing through points of the set ω remain in ω for all further values of variable t. Consequently, an n-parameter family of solutions of (1) $S^n(I) \subset \omega$. □

Lemma 2. *If, for the system (1), the scalar product $P > 0$ on W, then the system (1) has at least one solution on I whose graph belongs to the set ω.*

Proof. The condition $P > 0$ on W means that integral curves of the system (1) pass through points of the set W exiting from ω, i.e., the set W is a set of points of strict exit of integral curves of (1) with respect to the sets Ω and ω (see, e.g., [3, 5]).

Let Z be a set defined by the intersection of the set $\omega \cup W$ and the plane $t = t_1 \in I$, i.e., $Z = \{(x,t) \in \omega \cup W : t = t_1\}$. The set $Z \cap W$ is not a retract of Z, but is a retract of W. Hence, according to the retraction method , there is a point $P_1 \in Z \backslash W$, such that the integral curve of (1), passing through P_1, remains in ω for every $t \in I$, i.e., $K(P_1) \subset \omega$. Another intersection of $\omega \cup W$ with plane $t = t_2 \in I,\ t_2 \neq t_1$, should yield point P_2 and $K(P_2) \subset \omega$. However, those two integral curves can coincide, so we can only grant the existence of at least one solution of (1) on I whose graph belongs to the set ω . □

3. Main Results

Theorem 1. *Let $u \in C(I, \mathbb{R}_0)\,,\ \mathbb{R}_0 = [0, \infty)\,,$ such that*

$$\sum_{i=1}^{n} \alpha_i(t) \left| \sum_{j=1}^{n} a_{ij}(t)\,\varphi_j(t) + f_i(t) - \varphi_i'(t) \right| \le u(t)\,, \quad t \in I. \tag{2}$$

(i) If

$$\frac{1}{2} \sum_{j=1, j\neq i}^{n} \frac{1}{\alpha_i(t)\,\alpha_j(t)} \left| \alpha_i^2(t)\,a_{ij}(t) + \alpha_j^2(t)\,a_{ji}(t) \right| + u(t) < -a_{ii}(t) - \frac{\alpha_i'(t)}{\alpha_i(t)}, \quad i = 1, \ldots, n, \quad t \in I, \tag{3}$$

then the system (1) has a class of solutions $S^n(I) \subset \omega$.
(ii) If

$$\frac{1}{2} \sum_{j=1, j\neq i}^{n} \frac{1}{\alpha_i(t)\,\alpha_j(t)} \left| \alpha_i^2(t)\,a_{ij}(t) + \alpha_j^2(t)\,a_{ji}(t) \right| + u(t) \tag{4}$$

$$< a_{ii}(t) + \frac{\alpha_i'(t)}{\alpha_i(t)}, \quad i = 1, \dots, n, \quad t \in I,$$

then the system (1) has at least one solution on I whose graph belongs to the set ω.

Proof. For the scalar product P we have

$$P(x,t) = \sum_{i=1}^{n} \left\{ \alpha_i^2(t) [x_i - \varphi_i(t)] \left[\sum_{i=1}^{n} a_{ij}(t) x_j + f_i(t) \right] \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \alpha_i(t) \alpha_i'(t) [x_i - \varphi_i(t)]^2 - \alpha_i^2(t) \varphi_i'(t) [x_i - \varphi_i(t)] \right\}$$

$$= \sum_{i=1}^{n} \left\{ \alpha_i^2(t) [x_i - \varphi_i(t)] \left[\sum_{j=1}^{n} a_{ij}(t) [x_j - \varphi_j(t)] + \sum_{j=1}^{n} a_{ij}(t) \varphi_j(t) + f_i(t) \right] \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \alpha_i(t) \alpha_i'(t) [x_i - \varphi_i(t)]^2 - \alpha_i^2(t) \varphi_i'(t) [x_i - \varphi_i(t)] \right\}$$

$$= \sum_{i=1}^{n} \alpha_i^2(t) a_{ii}(t) [x_i - \varphi_i(t)]^2$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\alpha_i^2(t) a_{ij}(t) + \alpha_j^2(t) a_{ji}(t) \right] [x_i - \varphi_i(t)] [x_j - \varphi_j(t)]$$

$$+ \sum_{i=1}^{n} \left\{ \alpha_i^2(t) [x_i - \varphi_i(t)] \left[\sum_{j=1}^{n} a_{ij}(t) \varphi_j(t) + f_i(t) \right] \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \alpha_i(t) \alpha_i'(t) [x_i - \varphi_i(t)]^2 - \alpha_i^2(t) \varphi_i'(t) [x_i - \varphi_i(t)] \right\}$$

$$= \sum_{i=1}^{n} \left[\alpha_i^2(t) a_{ii}(t) + \alpha_i(t) \alpha_i'(t) \right] [x_i - \varphi_i(t)]^2$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\alpha_i^2(t) a_{ij}(t) + \alpha_j^2(t) a_{ji}(t) \right] [x_i - \varphi_i(t)] [x_j - \varphi_j(t)]$$

$$+ \sum_{i=1}^{n} \left\{ \alpha_i^2(t) [x_i - \varphi_i(t)] \left[\sum_{j=1}^{n} a_{ij}(t) \varphi_j(t) + f_i(t) - \varphi_i'(t) \right] \right\}$$

$$= \sum_{i=1}^{n} \left[a_{ii}(t) + \frac{\alpha_i'(t)}{\alpha_i(t)} \right] \alpha_i^2(t) [x_i - \varphi_i(t)]^2 +$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\alpha_i^2(t) a_{ij}(t) + \alpha_j^2(t) a_{ji}(t)}{\alpha_i(t) \alpha_j(t)} \alpha_i(t) [x_i - \varphi_i(t)] \alpha_j(t) [x_j - \varphi_j(t)]$$

$$+\sum_{i=1}^{n}\left\{\alpha_i^2(t)\left[x_i-\varphi_i(t)\right]\left[\sum_{j=1}^{n}a_{ij}(t)\varphi_j(t)+f_i(t)-\varphi_i'(t)\right]\right\}.$$

If we introduce a notation

$$X_i(x,t)=\alpha_i(t)\left[x_i-\varphi_i(t)\right],$$

$$A_{ii}(t)=a_{ii}(t)+\frac{\alpha_i'(t)}{\alpha_i(t)},\quad i=1,\dots,n, \tag{5}$$

$$A_{ij}(t)=\frac{1}{2\alpha_i(t)\alpha_j(t)}\left[\alpha_i^2(t)a_{ij}(t)+\alpha_j^2(t)a_{ji}(t)\right],\ i\neq j\ \ (A_{ij}=A_{ji}), \tag{6}$$

$$R(x,t)=\sum_{i=1}^{n}\alpha_i(t)X_i(x,t)\left[\sum_{j=1}^{n}a_{ij}(t)\varphi_j(t)+f_i(t)-\varphi_i'(t)\right],$$

we have

$$P(x,t)=\sum_{i=1}^{n}A_{ii}(t)X_i^2(x,t)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}2A_{ij}(t)X_i(x,t)X_j(x,t)+R(x,t).$$

Now, according to the conditions (2), (3) and (4) and the inequality $ab\leq\frac{1}{2}\left(a^2+b^2\right)$ $\forall a,b\in\mathbb{R}$, the following estimates for P on W are valid:

(i)

$$\begin{aligned}P(x,t) &\leq \sum_{i=1}^{n}A_{ii}(t)X_i^2(x,t)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|A_{ij}(t)\right|\left[X_i^2(x,t)+X_j^2(x,t)\right]\\ +\left|R(x,t)\right| &= \sum_{i=1}^{n}\left[A_{ii}(t)+\sum_{j=1,j\neq i}^{n}\left|A_{ij}(t)\right|\right]X_i^2(x,t)+\left|R(x,t)\right|\\ &< \sum_{i=1}^{n}\left[-u(t)\right]X_i^2(x,t)+\left|R(x,t)\right|=-u(t)+\left|R(x,t)\right|\leq 0,\end{aligned}$$

(ii)

$$\begin{aligned}P(x,t) &\geq \sum_{i=1}^{n}A_{ii}(t)X_i^2(x,t)-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|A_{ij}(t)\right|\left[X_i^2(x,t)+X_j^2(x,t)\right]\\ -\left|R(x,t)\right| &= \sum_{i=1}^{n}\left[A_{ii}(t)-\sum_{j=1,j\neq i}^{n}\left|A_{ij}(t)\right|\right]X_i^2(x,t)-\left|R(x,t)\right|\\ &> \sum_{i=1}^{n}u(t)X_i^2(x,t)-\left|R(x,t)\right|=u(t)-\left|R(x,t)\right|\geq 0.\end{aligned}$$

The statements obtained for the function P, according to the given *Lemmas*, grant the presented statement. $\square$

Theorem 2. *Let $u, v_i \in C(I, \mathbb{R}_0)$, $i = 1, \ldots, n$, such that*

$$\alpha_i(t)\left|\sum_{j=1}^{n} a_{ij}(t)\varphi_j(t) + f_i(t) - \varphi_i'(t)\right| \le v_i(t), \quad \sum_{i=1}^{n} v_i^2(t) \le u^2(t), \qquad i = 1, \ldots, n. \tag{7}$$

(i) Under condition (3) the system (1) has a class of solutions $S^n(I) \subset \omega$.
(ii) Under condition (4) the system (1) has at least one solution on I whose graph is contained in the set ω.

Proof. Using the proof of *Theorem 1* it is sufficient to note that in view of (7), $|R(x,t)| \le u(t)$ on W. Indeed, using the concept of the relative extremum of the function $|R(x,t)|$ on W, we have

$$\begin{aligned} |R(x,t)| &\le \sum_{i=1}^{n} \alpha_i(t)\left|\sum_{j=1}^{n} a_{ij}(t)\varphi_j(t) + f_i(t) - \varphi_i'(t)\right| |X_i(x,t)| \\ &\le \sum_{i=1}^{n} v_i(t)|X_i(x,t)| \le \left(\sum_{i=1}^{n} v_i^2(t)\right)^{\frac{1}{2}} \le u(t).\square \end{aligned}$$

4. Applications

Let us note that conditions of *Theorems* 1 and 2 are simplified if curve Γ is some particular curve and if we take the function $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ in an appropriate special form, e.g., $\alpha_i(t) = \beta(t)$ or $\alpha_i(t) = q = const$, $i = 1, \ldots, n$.

Theorem 3. *Let Γ be the curve of stationary points of (1) and let $u \in C(I, \mathbb{R}_0)$ such that*

$$\sum_{i=1}^{n} [\alpha_i(t)\varphi_i'(t)]^2 \le u^2(t), \quad t \in I. \tag{8}$$

(i) If (3) holds true, the system (1) has a class of solutions $S^n(I) \subset \omega$.
(ii) If (4) holds true, the system (1) has at least one solution on I whose graph belongs to the set ω.

Proof. It is sufficient to note that the condition (8) follows from (7) when Γ is the curve of stationary points of the system (1). $\square$

Example 1. *The system of equations*

$$\begin{aligned} x_1' &= \left(1 - t^3\right)x_1 + t^3 x_2 \sin t - \sin t, \\ x_2' &= \left(2 - t^3 \sin t\right)x_1 - t^3 x_2 + t^3\left(1 + \sin^2 t\right) - 2\sin t \end{aligned} \tag{9}$$

has a two-parameter family of solutions $x(t) = (x_1(t), x_2(t))$ satisfying the condition

$$[x_1(t) - \sin t]^2 + [x_2(t) - 1]^2 < t^{-4} \quad \forall t > 2. \tag{10}$$

Here we consider the behaviour of solutions of (9) with respect to the set

$$\omega_1 = \left\{ (x_1, x_2, t) \in \mathbb{R}^3 : (x_1 - \sin t)^2 + (x_2 - 1)^2 < t^{-4},\ t > 2 \right\},$$

i.e., in $\delta-$neighbourhood, $\delta(t) = t^{-2}$, of the curve of stationary points $(\sin t, 1, t)$, $t > 2$, of the system (9). Here we have $\alpha(t) = \left(t^2, t^2\right)$ and $u(t) = t^2$.

Theorem 4. *Let* Γ, *be an integral curve of the system (1).*
(i) If

$$\frac{1}{2} \sum_{j=1, j \neq i}^{n} \frac{1}{\alpha_i(t)\,\alpha_j(t)} \left| \alpha_i^2(t)\, a_{ij}(t) + \alpha_j^2(t)\, a_{ji}(t) \right| < -a_{ii}(t) - \frac{\alpha_i'(t)}{\alpha_i(t)},\ i = 1, \ldots, n, \quad t \in I, \tag{11}$$

then the system (1) has a class of solutions $S^n(I) \subset \omega$.
(ii) If

$$\frac{1}{2} \sum_{j=1, j \neq i}^{n} \frac{1}{\alpha_i(t)\,\alpha_j(t)} \left| \alpha_i^2(t)\, a_{ij}(t) + \alpha_j^2(t)\, a_{ji}(t) \right| < a_{ii}(t) + \frac{\alpha_i'(t)}{\alpha_i(t)},\ i = 1, \ldots, n, \quad t \in I, \tag{12}$$

then the system (1) has at least one solution on I *whose graph belongs to the set* ω.

Proof. Here is

$$\sum_{j=1}^{n} a_{ij}(t)\,\varphi_j(t) + f_i(t) - \varphi_i'(t) \equiv 0 \text{ on } I,\ i = 1, \ldots, n,$$

and the proof of this Theorem follows from the proof of *Theorem 1* for $u(t) = 0$. □

Theorem 5. *Let* Γ *be an integral curve of the system (1) and let terms* $A_{ij}(t)$ *be defined by (5) and (6).*
(i) If

$$(-1)^k \det \begin{pmatrix} A_{11}(t) \ldots A_{1k}(t) \\ \vdots \\ A_{k1}(t) \ldots A_{kk}(t) \end{pmatrix} > 0, \quad k = 1, \ldots, n, \quad t \in I, \tag{13}$$

then the system (1) has a class of solutions $S^n(I) \subset \omega$.
(ii) If

$$\det \begin{pmatrix} A_{11}(t) \ldots A_{1k}(t) \\ \vdots \\ A_{k1}(t) \ldots A_{kk}(t) \end{pmatrix} > 0, \quad k = 1, \ldots, n, \quad t \in I, \tag{14}$$

then the system (1) has at least one solution on I *whose graph is contained in the set* ω.

Proof. In this case the function of a scalar product P on the corresponding set W is

$$\begin{aligned} P(x,t) &= \sum_{i=1}^{n} A_{ii}(t) X_i^2(x,t) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2A_{ij}(t) X_i(x,t) X_j(x,t) \\ &= \sum_{i,j=1}^{n} A_{ij}(t) X_i(x,t) X_j(x,t), \end{aligned}$$

i.e., P is a quadratic symmetric form. According to the Sylvester criterion condition (13) grants that $P<0$, and condition (14) grants that $P>0$ on the corresponding set W. This, according to given *Lemmas*, confirms the statements of the *Theorem 5.* □

Now, let us consider the system (1) in which coefficients $a_{ij}(t) \equiv 0$ for $|i-j| > 1$, i.e., system

$$\begin{gathered} x_1' = a_{11}(t) x_1 + a_{12}(t) x_2 + f_1(t), \\ x_i' = a_{i,i-1}(t) x_{i-1} + a_{ii}(t) x_i + a_{i,i+1}(t) x_{i+1} + f_i(t), \quad i = 1, \dots, n-1, \\ x_n' = a_{n,n-1}(t) x_{n-1} + a_{nn}(t) x_n + f_n(t), \end{gathered} \tag{15}$$

where coefficients satisfy the condition (for every $i \in \{2, \dots, n\}$)

$$a_{i-1,i}(t)\, a_{i,i-1}(t) < 0, \quad t \in I \tag{16}$$

or

$$a_{i-1,i}(t) = a_{i,i-1}(t) = 0, \quad t \in I. \tag{17}$$

Let us define positive functions $\eta_k,\ k = 1, \dots, n,$

$$\eta_1(t) = 1,\ \eta_k^2(t) = \eta_{k-1}^2(t) \left| \frac{a_{k-1,k}(t)}{a_{k,k-1}(t)} \right|,\ k = 2, \dots, n. \tag{18}$$

If condition (17) is valid for a certain i, then we should take $\eta_i(t) = 1$.

Corollary 1. *Let* $\mu \in C^1(I, \mathbb{R}^+)$ *and*

$$\alpha_i(t) = \mu(t)\, \eta_i(t),\ i = 1, \dots, n,$$

where functions η_i *are defined by (18) and let* Γ *be an integral curve of the system (15).*

(i) If

$$a_{ii}(t) + \frac{\alpha_i'(t)}{\alpha_i(t)} < 0,\ i = 1, \dots, n,\ t \in I,$$

then the system (15) has a class of solutions $S^n(I) \subset \omega$.

(ii) If

$$a_{ii}(t) + \frac{\alpha_i'(t)}{\alpha_i(t)} > 0,\ i = 1, \dots, n,\ t \in I,$$

then the system (15) has at least one solution on I *whose graph belongs to the set* ω.

Proof. It is sufficient to note that

$$\alpha_k^2(t)\,a_{k,k-1}(t)+\alpha_{k-1}^2(t)\,a_{k-1,k}(t)=0,\ k=2,\dots,n,\ \forall t\in I$$

and conditions of the *Theorem 4* are evidently satisfied. □

Example 2. *The system of equations*

$$\begin{aligned} x_1' &= \frac{x_1}{t^2}\sin t+g(t)\,x_2+f_1(t),\\ x_2' &= -g(t)\,x_1+\frac{1}{t^2}x_2-t^2x_3+f_2(t),\\ x_3' &= 4(t+1)^2x_2+\frac{1}{2t^2+\sin t}x_3+f_3(t), \end{aligned} \tag{19}$$

where $g, f_i\in C(I,\mathbb{R})$, *has a tree-parameter family of solutions* $x(t)=(x_1(t),x_2(t),x_3(t))$ *satisfying the condition*

$$\begin{aligned} &4\left(1+\frac{1}{t}\right)^2\left\{[x_1(t)-\varphi_1(t)]^2+[x_2(t)-\varphi_2(t)]^2\right\}+[x_3(t)-\varphi_3(t)]^2\\ <\ &C^2\left(1-\frac{1}{t+1}\right)^2\ \forall t>2 \end{aligned}$$

where $\varphi(t)=(\varphi_1(t),\varphi_2(t),\varphi_3(t))$ *is an arbitrary solution of the system (19),* $C>0$ *is an arbitrary constant.*

In the case of the system (19) we have

$$\eta_1(t)=\eta_2(t)=1,\ \eta_3(t)=\frac{t}{2(t+1)}\text{ and }\mu(t)=\frac{2}{C}\left(1+\frac{1}{t}\right)^2.$$

Remark 1. *The obtained results also contain an answer to the question on approximation of solutions* $x(t)$ *whose existence is established. The errors of the approximation are defined by the function* $\delta(t)=\frac{1}{\alpha(t)}$ *which can be arbitrarily small* $\forall t\in I$. *In case when the function* $\delta(t)$ *tends to zero as* $t\to\infty$ *we have asymptotic solutions of the corresponding system of differential equations.*

In *Example 1* we have $\delta(t)=\left(t^{-2},t^{-2}\right)$ and, for a sufficiently large t, each curve $K\subset\omega_1$ can be taken as an approximate integral curve of the system (15), for example $x(t)=(\sin t,1)$ is an approximate solution and the inequality (10) presents an estimate of the accuracy of this approximate solution.

In *Example 2* we have $\delta(t)=\left(\frac{C}{2}\left(1-\frac{1}{t+1}\right)^2,\frac{C}{2}\left(1-\frac{1}{t+1}\right)^2,C\left(1-\frac{1}{t+1}\right)\right)$, where $C>0$ can be arbitrarily small.

Remark 2. *The obtained results give the possibility that we can talk about the stability of the system (1). For example, in view of the Theorem 4, in case (i) the system (1) is stable, and in case (ii) the system (1) is unstable according to Ljapunov, only if the conditions (11) and (12) are valid for* $\alpha(t)=q=const$.

The system (19) is stable, and the system (9) is asymptotically stable.

Remark 3. *Let us have a quasilinear system*

$$x'=A(x,t)\,x+f(x,t), \tag{20}$$

where a matrix-function $A(x,t)$ and a vector-function $f(x,t)$ satisfy the conditions sufficient for the existence and uniqueness of solutions which are determined by initial data in Ω. The obtained results for the system (1) are also valid for the system (20) only if $A(x,t)$ and $f(x,t)$ satisfy the corresponding conditions on $\bar{\omega}$.

Example 3. *Consider the system of Volterra type*

$$x' = [p_i(x,t) - f_i(x,t)]\, x_i, \quad i = 1, \ldots, n, \tag{21}$$

where $p_i, f_i \in C^1(\Omega, \mathbb{R})$.

Corollary 2. *Let $p \in \mathbb{R}_0,\ q \in \mathbb{R}^+$ and*

$$\omega_3 = \left\{ (x,t) \in \Omega : \sum_{i=1}^{n} x_i^2 < q^2 e^{-2pt} \right\}.$$

If

$$p_i(x,t) - f_i(x,t) < -p \leq 0,\ i = 1, \ldots, n \quad \text{on } \partial\omega_3 \tag{22}$$

or

$$p_i(x,t) < -p \leq 0,\ \sum_{i=1}^{n} x_i^2 f_i(x,t) \geq 0,\ i = 1, \ldots, n \quad \text{on } \partial\omega_3, \tag{23}$$

then the system (21) has $S^n(I) \subset \omega_3$ and the system (21) is exponential asymptotically stable.

Proof. Here we consider the behaviour of solutions of (21) with respect to the set ω_3. The system (21) is exponential asymptotically stable because the conditions (22) and (23) are valid with the function qe^{-pt} for every $q > 0$. □

Takeuchi [7] considers the stability of the solution $x = 0$ of the system

$$x' = [p_i - f_i(x)]\, x_i, \quad i = 1, \ldots, n,\ p_i \in \mathbb{R}.$$

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